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Genetic algorithms for solving portfolio allocation models based on relative-entropy, mean and variance

Ozkan Aslan [1], Yeliz Mert Kantar [2,*], Ilhan Usta [2]

[1]*Department of Computer Engineering, Engineering Faculty, Anadolu University, Eskisehir, Turkey*
[2]*Department of Statistics, Science Faculty, Eskisehir, Turkey*

Abstract: In this study, we consider relative entropy for the problem of the portfolio selection, by taking into accounting the well-known objectives such as the mean and variance–covariance matrix of the asset returns. The resulting non-linear optimization problem for the model based on relative entropy, mean and variance has been solved with the genetic algorithms. The genetic algorithm is one of the alternative methods to solve such problems since it produces acceptable solutions within a reasonable time for optimization problems which have a lot of constraints and a lot of objective functions. The performance of the considered model is illustrated with an application to various assets taken from the Borsa Istanbul. Analysis shows that as well as genetic algorithm can effectively solve the optimization problem formed by relative entropy, mean and variance, also this model performs well relative to Markowitz's mean-variance model for out of sample cases.

Key words: Portfolio selection; Entropy; Relative entropy; Genetic algorithm

1. Introduction

Determination of portfolio is one of the most challenging problems in finance field. Assigning the weights to assets in a portfolio requires the well-defined objective functions. Markowitz's variance has been accepted as objective function for portfolio allocation problem (Markowitz, 1952). This objective function together with the expectation of portfolio, which is called as mean-variance model (MVM), has been widely used to determine portfolio weights (probabilities).

MVM has been generally used as the minimization of the portfolio variance for a given level of return or the maximization of the portfolio return for a given level of variance (Sefiane and Benbouziane, 2012). However, it is well-known that mean and variance are generally inadequate to explain portfolios in the case of non-normal return distribution (Chunhachinda et al., 1997; Lai, 1991), moreover, the portfolio weights obtained from MVM can often focus on a few assets or extreme positions and also MVM does not work well for the out-of-sample cases (Park and Bera, 2008). On the other hand, some studies have pointed that one of objectives of portfolio selection is to reduce unsystematic risk in portfolio. Unsystematic risk is also called "diversifiable risk" since diversification can minimize the risk of owning any single investment. In other words, "Diversification can nearly eliminate unsystematic risk". In order to ensure the diversification in portfolio, entropy function can be used. It is well-known that the greater the value of entropy measure for portfolio weights, the highest the portfolio diversification is. In literature, (Bera and Park, 2008, 2005) and Usta and Kantar (2008, 2011) present the asset allocation models based on entropy and relative entropy measures in order to generate a well-diversified portfolio.

On the other hand, another issue is to be considered is inclusion of knowledge of specialists or managers into portfolio model. However, it is hard to both achieve a portfolio determined by a specialist and satisfy the objectives of portfolio such as variance and expectation. Relative entropy (RE) or Kullback-Leibler is a measure defined as directed divergence distance between two distributions. Based on this measure, Kullback's minimum cross-entropy (MinxEnt) principle was proposed as a method in order to determine probability density function of random variable when information is given as moments. This principle ensures finding a distribution with the least relative entropy measure with respect to the pre-determined prior distribution at the same time, while it is satisfying the moment conditions. When it is taken into account various objectives such as mean, variance and RE for portfolio selection, the resulting model is to be solved by numerical methods. There is huge literature for such numerical methods for portfolio models.

In this study, as different from previous studies, we study the genetic algorithms (GA) for the optimization model based on RE, mean and variance. This model is denoted by REMVM. GAs are based on

* Corresponding Author.

the natural selection principles (Bozdoğan, 2004) and (Gharehchopogh and Maroufi, 2014), (Gharehchopogh and Pourali, 2015) and “they can deal with nonlinear optimization problems with non-smooth and even non-continuous objective, and continuous and/or integer variables” (Dan et al. 2005). However, the choice of GA parameters such as the mutation and crossover can influence the performance of GA. For these reason, GA is a heuristic and stochastic method as distinct from classical optimization techniques. It overcomes the local optimum problem by synthesizing the best solutions. Moreover, it produces nearly good solutions in a short time for the problems which are hard to solve with other optimization methods (Masini and Speranza, 1999). Also, as well as the solution of REMVM with GA, we also consider performance of REMVM relative to MVM for out-of sample cases with an application to the various assets taken from the Borsa Istanbul 100 index (BIST 100) as in (Usta and Kantar, 2011). Analysis shows that the REMVM generally produces more diversified portfolio than classical MVM if the prior distribution is taken as equally weight (1/n). Also, the REMVM model performs well relative to MVM for out-of sample cases. Considering all these issues, this study is organized as follows: traditional portfolio selection models are presented briefly in Section 2. Relative entropy and some information measures are discussed in Section 3. Formulation of portfolio selection model for GA is presented in Section 4. The process of GA is provided in Section 5. Next, an empirical study is conducted the in Section 6. Finally, conclusions and suggestions are presented in Section 6.

2. Traditional portfolio selection models

Portfolio is the collection of assets held by an investor or a company. Therefore, the portfolio selection problem is to determine the collection of assets by means of optimum strategy. Each possible strategy corresponding to certain objective or objectives is considered as a portfolio selection model. In this section, we provide necessary definitions and notations which are used in portfolio selection model. The vector of portfolio weights is $\boldsymbol{w} = (w_1, \ldots, w_n)'$, where w_i is the weight of ith asset in the portfolio. The portfolio weights satisfy $\sum_{i=1}^{n} w_i = 1$ and also $0 \le w_i \le 1$ is generally desired if short selling is not allowed. The vector of returns is $\boldsymbol{R} = (R_1, \ldots, R_n)$ and the mean vector of returns is $\boldsymbol{M} = E(\boldsymbol{R}) = (M_1, \ldots, M_n)$, the nxn variance-covariance matrix of returns is $E(\boldsymbol{R}) = E[(\boldsymbol{R} - E(\boldsymbol{R}))^2] = \boldsymbol{V}$ where $\boldsymbol{V}$ consists of elements of σ_{ij}, which show covariance between the return of asset i and j for every i and j.

Mean, variance and standard deviation of the returns of portfolio are respectively given as follows:

$$E(\boldsymbol{R_p}) = E[\boldsymbol{w}'\boldsymbol{R}] = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} w_i M_i = \boldsymbol{w}'\boldsymbol{M} \quad (1)$$

$$Var(\boldsymbol{R_p}) = Var[\boldsymbol{w}'\boldsymbol{R}] = \boldsymbol{w}'\boldsymbol{V}\boldsymbol{w} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} w_i w_j \sigma_{ij} \quad (2)$$

$$sd(\boldsymbol{R_p}) = \sqrt{\boldsymbol{w}'\boldsymbol{V}\boldsymbol{w}} \quad (3)$$

As an alternative to the standard deviation of portfolio, mean absolute deviation (MAD) given in (4) is proposed by Konno and Yamazaki (1992).

$$MAD(\boldsymbol{R_p}) = \boldsymbol{E}[|\boldsymbol{R} - E(\boldsymbol{R})|] \quad (4)$$

However, it shown that MAD is theoretically the same with the variance of variance when the returns of assets are normally distributed. Only difference is that the solution of model based on MAD is easier than MVM.

3. Relative entropy measure

The measures of distance between probability distributions have been used in the problems of inference, similarity and discrimination. Various types of distance measures have been proposed in the literature (Ullah, 1996). The divergence measures based on the idea of information theory first was introduced in (Shannon, 1948). Shannon proposed entropy as uncertainty measure. Then, Kullback introduced the relative entropy measure between two statistical distributions. Based on this entropy and RE measures, the maximum entropy (MaxEnt) principle and minimum cross entropy (MinxEnt) principle are proposed to determine the probability distribution of a random variable. In recent years, various versions of the Shannon entropy and RE measures have been seen in the literature, (Ullah, 1996). The Shannon’s entropy and RE measures for discrete random variable are respectively given as follows:

$$H(w) = -\sum_{i=1}^{n} w_i \log(w_i) \quad (5)$$

$$RE(w) = \sum_{i=1}^{n} w_i \log(\frac{w_i}{q_i}) \quad (6)$$

H (w) reaches its maximum value, when $w_i = 1/n$. Due to this properties of entropy measure, $H(w)$ provides a good measure of diversity for portfolio (Bera and Park, 2008) and (Usta and Kantar, 2011). RE is nonnegative and reaches its minimum value 0 if and only if $w_i = q_i$. Also Golan et al. (1996) show that

$$RE(w:q) = \sum_{i=1}^{n} w_i \log(\frac{w_i}{q_i}) = \frac{1}{q_i}\sum_{i=1}^{n}(w_i - q_i)^2 \quad (7)$$

Portfolio selection problem can be considered as determination of the distribution of the discrete variable. The probabilities are considered as portfolio weights.

4. Formulation of portfolio selection model

As is well known, the common approach of portfolio selection problem is MVM proposed by Markowitz (1952). However, this approach does not take into account the pre-determined portfolio which was determined by expert or manager.

Moreover, the portfolios constructed from MVM are often extremely concentrated on a few assets. Relative entropy measure can be used to integrate the MVM model with the pre-determined distribution. If the pre-determined distribution is taken as equally portfolio, the maximum diversification can be achieved. Pre-determined distribution can be obtained from investor or the knowledge of specialist. The considered portfolio optimization model can be given as follows:

Minimize

$$RE(w) = \sum_{i=1}^{n} w_i \log(\frac{w_i}{q_i}) \quad (8)$$

Where $q = (q_1, \dots, q_n)$ is the pre-determined distribution.

$$Var(\boldsymbol{R_p}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \leq V_{min} \quad (9)$$

$$E(\boldsymbol{R_p}) = \sum_{i=1}^{n} w_i m_i \geq E_{max} \quad (10)$$

$$\sum_{i=1}^{n} w_i = 1 \quad (11)$$

$$0 \leq w_i \leq 1, \quad (12)$$

In order to find the minimum of equations (8), subject to constraints (9), (10), (11) and (12), GA can be used. In the next section we will provide the basic principle and terminology of genetic algorithms.

5. Genetic algorithm

Genetic algorithm (GA) is inspired by the evolution theory. GA is a problem solving technique proposed by Holland (1975). GA is a heuristic and probabilistic method differently from classic optimization techniques. Although GA is generally used in optimization problems, it is also applied to many areas such as robotic, image processing, data analysis etc. Particularly, it is mostly preferred in estimating parameter that has large space and a lot of local optimum, in comparison with conventional methods.

The natural selection principle is at the heart of GA. Since GA can produce new solutions by synthesizing good candidates and applying mutations, it overcomes the local optimum problem. Furthermore, it propounds tolerable solutions in a short time for NP-complete that cannot be solved.

GA has roughly five steps: (1) generating the first generation randomly, (2) calculating the fitness value, (3) selection, (4) crossing-over, (5) mutation. If the stop-criterion is not achieved, iteration continues. In step (1), a sample from universe (search space) that includes possible solutions of problem establishes the first candidate solutions in the population. In step (2), individuals in the population are evaluated for a criterion named fitness value. Fitness value is calculated by fitness function, which is a component that varies from problem to problem. In step (3), parents are selected with regard to their fitness values. Various selection ways are used in this step such as roulette wheel selection and tournament selection. In step (4), selected parents are intercrossed in order. Simply, some genes are selected from one parent and other genes are selected from other parent. Thus a new individual comes up. This person has gene sequences such that any gene is not empty and number of genes is equal to the parents. In step (5), a small number of generated individuals changes randomly. Mutation succeeds if it is implemented slightly. It must be defined some basic constraints in GA in order to provide the optimization equation given equations (8-12).

Penalty-1: Sum of weights is 1.

$$f(w) = \begin{cases} 0, & \sum_{i=1}^{n} w_i = 1 \\ 2^{|(\sum_{i=1}^{n} w_i)-1|}, & otherwise \end{cases}$$

Penalty-2: Weights are not negative.

$$g(w) = \sum_{i=1}^{n} sgn((sgn(w_i) - 1)^2)2^{-w_i}$$

Thus, the equation of optimization for minimization is follows:

$$O(w) = f(w) + g(w) + Var(\boldsymbol{R_p}) + \frac{1}{\beta}RE - E(\boldsymbol{R_p})$$

6. The empirical study

In this section, we give the descriptions of empirical datasets used in this study and present the results of empirical study.

6.1. Data description

Data set includes monthly returns of 5, 10 and 15 assets which are traded in the BORSA Istanbul in Turkey, from different sectors: Financial Institutions, Manufacturing Industry and Technology sectors. The dataset are taken from Borsa Istanbul web site. The period for dataset is from January 1999 to September 2012 (monthly observations).

The descriptive statistics results for these assets are given in Table 1, Table 2 and Table 3, respectively.

Table 1: Descriptive statistics and normality test results for the returns of five assets.

Portfolio	Mean	Variance	Skewness	Kurtosis	JB
X1	0.0367	0.0411	0.5451	3.7252	0
X2	0.0307	0.0582	-0.0397	3.1950	0
X3	0.0254	0.0551	0.5939	3.7888	0
X4	0.0157	0.0704	-0.3980	3.9875	0
X5	0.0629	0.0701	0.8856	4.2925	1

Note: JB is the value of the Jarque–Bera test for normality. 0 and 1 denotes normality and non-normality, respectively

The statistics in Table 1, 2 and 3 give some insight into the characteristics of return data. As can be seen from these Tables, empirical datasets exhibit different statistical characteristics, since they have different degrees of skewness and kurtosis. Also, Jack-Bera test for some of return distribution of three empirical datasets reject the null hypothesis for normality at the 5% significance level.

Table 2: Descriptive statistics and normality test results for the returns of ten assets.

Portfolio	Mean	Variance	Skewness	Kurtosis	JB
X1	0.0368	0.0411	0.5451	3.7252	0
X2	0.0307	0.0582	-0.0397	3.1950	0
X3	0.0254	0.0551	0.5939	3.7888	0
X4	0.0157	0.0704	-0.3980	3.9875	0
X5	0.0629	0.0701	0.8856	4.2925	1
X6	0.0447	0.0607	-0.0487	2.1373	0
X7	0.0212	0.0652	1.2035	6.1877	1
X8	0.0358	0.0429	0.6965	3.3060	0
X9	0.0260	0.0866	-0.0588	3.3643	0
X10	0.0405	0.0396	0.1104	3.1023	0

Table 3: Descriptive statistics and normality test results of 15 assets

Portfolio	Mean	Variance	Skewness	Kurtosis	JB
X1	0.0368	0.0411	0.5451	3.7252	0
X2	0.0307	0.0582	-0.0397	3.1950	0
X3	0.0254	0.0551	0.5939	3.7888	0
X4	0.0157	0.0704	-0.3980	3.9875	0
X5	0.0629	0.0701	0.8856	4.2925	1
X6	0.0447	0.0607	-0.0487	2.1373	0
X7	0.0212	0.0652	1.2035	6.1877	1
X8	0.0358	0.0429	0.6965	3.3060	0
X9	0.0260	0.0866	-0.0588	3.3643	0
X10	0.0405	0.0396	0.1104	3.1023	0
X11	0.0397	0.0542	0.2368	2.6381	0
X12	0.0181	0.0333	0.4768	3.1426	0
X13	0.0288	0.0513	0.3191	3.8493	0
X14	0.0181	0.0619	0.4594	4.4395	1
X15	0.0415	0.0691	0.3679	3.0706	0

Firstly, we have done some experiment to observe whether REMVM leads to well-diversified portfolio when pre-determined portfolio is taken as 1/n. For this reason, we consider 5, 10 and 15 assets. In figures, BASE denotes equal weights, 1/n.

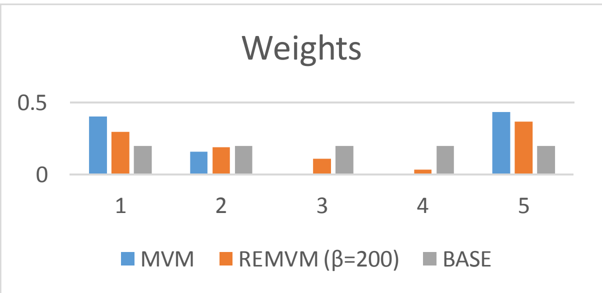

Fig. 1: Weights obtained from models for 5 assets.

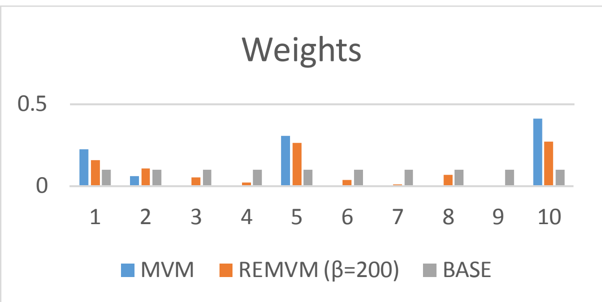

Fig. 2: Weights obtained from models for ten assets.

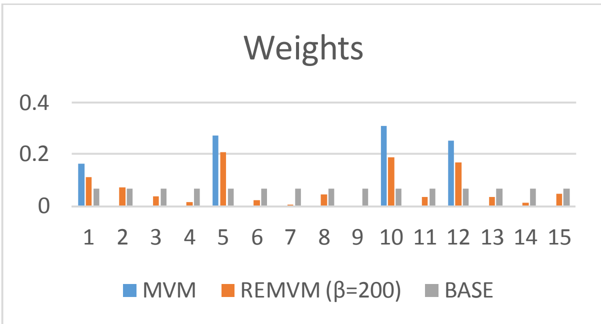

Fig. 3: Weights obtained from models for 15 assets

Also the weights obtained from models are given in Fig. 1, 2, 3. As seen from figures, the MVM focuses on few assets. In other words, the MVM leads to less diversification compared to the REMVM. REMVM yields more diversified portfolio than MVM. This is preferred case by most of the asset managers.

In order to observe the performance of REMVM versus MVM, we provide return, variance (risk) and Sharpe-ratio (SR) for in-sample and out-of sample cases in Table 4, 5 and 6. It is obvious from these Tables that MVM performs better than the other models for all in-sample cases, as expected. This is natural result since MVM depends on return and variance, thus Sharpe ratio (SR). For this reason, we focus on out-of sample performance of the considered models. In order to observe out-of sample performance of REMVM versus MVM, portfolio weights are calculated from first 12 months as a training set, Risk (variance), Return and SR are calculated with the determined weights over the next 3, 6, 9 months as test set. We repeat this procedure by dropping first month, the mean values of the obtained returns, risk, SR values for 3, 6, 9 months are provided in Tables. In Table 4, 5 and 6, by taking into account the importance of RE measure within the context of REMVM, β is taken as 1, 2, 20 and 200. Thus, each β value represents distinct portfolio weights. It should be noted that when β is equal to 200, thus, 1/β =0.005, it means that the importance of RE is the least. Therefore, REMVM is approximately equal to MVM, as observed in Tables. It is observed that the out-of-sample performances of REMVM are superior to classical MVM according to Return, risk, and SR.

Table 4: In- and out-of-sample performance of models for 5 assets

	tseT	MVM	$REMVM_1$	$REMVM_2$	$REMVM_3$	$REMVM_4$
	Return	0.04421	0.04364	0.03292	0.02280	0.02196
IS	Risk	0.04922	0.04920	0.05441	0.05988	0.06039
	SR	0.21523	0.21398	0.15898	0.10937	0.10549
	Return	0.01166	0.01194	0.01165	0.01155	0.01155
3	Risk	0.01460	0.01411	0.01331	0.01357	0.01361
	SR	0.27977	0.32687	0.50438	0.46037	0.44375
	Return	0.00952	0.00972	0.01052	0.01138	0.01149
6	Risk	0.01674	0.01637	0.01558	0.01594	0.01600
	SR	0.16775	0.17124	0.19689	0.20659	0.20691
	Return	0.00900	0.00905	0.01035	0.01211	0.01230
9	Risk	0.01785	0.01743	0.01632	0.01672	0.01678
	SR	0.15695	0.15698	0.17198	0.18388	0.18471

Note: $REMVM_1$, $REMVM_2$, $REMVM_3$, $REMVM_4$ are corresponding to β=200, β=20, β=2, β=1 and IS means as in-sample performance and 3, 6, 9 show the period of 3, 6 and 9 months for out-of sample performance, respectively in Table 4,5,6.

Table 5: In- and out-of-sample performance of models for 10 assets

	tseT	MVM	$REMVM_1$	$REMVM_2$	$REMVM_3$	$REMVM_4$
	Return	0.04933	0.04841	0.03282	0.02041	0.01956
IS	Risk	0.04979	0.04960	0.05788	0.06712	0.06789
	SR	0.23942	0.23672	0.15491	0.09329	0.08934
	Return	0.00656	0.00722	0.00930	0.00871	0.00876
3	Risk	0.01395	0.01317	0.01207	0.01231	0.01234
	SR	0.21105	0.23034	0.27990	0.29745	0.30176
	Return	0.00559	0.00616	0.00858	0.00867	0.00875
6	Risk	0.01640	0.01572	0.01430	0.01452	0.01454
	SR	0.14681	0.15153	0.17791	0.17462	0.17632
	Return	0.00650	0.00706	0.00885	0.00937	0.00946
9	Risk	0.01759	0.01687	0.01504	0.01520	0.01522
	SR	0.14229	0.14554	0.15813	0.15368	0.15381

Table 6: In- and out-of-sample performance of models for 15 assets

	tseT	MVM	$REMVM_1$	$REMVM_2$	$REMVM_3$	$REMVM_4$
	Return	0.05490	0.05339	0.03348	0.02041	0.01963
IS	Risk	0.04857	0.04792	0.05682	0.06712	0.06687
	SR	0.26165	0.25740	0.15461	0.09329	0.08863
	Return	0.00587	0.00728	0.00928	0.00909	0.00923
3	Risk	0.01244	0.01193	0.01143	0.01200	0.01205
	SR	0.34521	0.35382	0.48731	0.49977	0.52372
	Return	0.00386	0.00516	0.00853	0.00908	0.00921
6	Risk	0.01485	0.01415	0.01338	0.01400	0.01403
	SR	0.18395	0.20935	0.21270	0.19198	0.19132
	Return	0.00390	0.00495	0.00876	0.00981	0.00991
9	Risk	0.01610	0.01529	0.01409	0.01463	0.01465
	SR	0.14546	0.15816	0.17825	0.16942	0.16901

7. Conclusions

The main results obtained from the presented study can be listed as follows:

1. The model based on RE, mean and variance (REMVM) has been considered for portfolio selection problem with GA.
2. The resulting nonlinear model is solved with GA.
3. It is observed that GA can solve the optimization problem (REMVM) efficiency.

4. It is observed that the importance of the objective functions in portfolio allocation model can be adjusted thanks to GA.
5. REMVM is tested on the various assets taken from the BIST 100. It is observe that REMVM model performs well relative to MVM model for out-of sample cases.

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